The finite element discretization error estimate and $H^1$ regularity are shown for the solution generated by Newton’s method to the stationary compressible Navier-Stokes equations by interpreting Newton’s method as an equivalent iterative method. © 2003 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 19: 511–524, 2003

Keywords: compressible Stokes equations; Newton’s method

I. INTRODUCTION

Let $\Omega$ be a bounded, open domain in $\mathbb{R}^2$ with a $C^2$ boundary $\Gamma$. Consider the system of equations of the stationary compressible Navier-Stokes equations for the velocity $u = (u_1, u_2)'$ and the pressure $p$ (cf: [1, 2]) as follows:

\[
\begin{aligned}
-\mu \Delta u - \nu \nabla \cdot u + (u \cdot \nabla)u + \nabla p &= f & & \text{in } \Omega, \\
\nabla \cdot u + \beta \cdot \nabla p &= g & & \text{in } \Omega, \\
\n\quad u &= 0 & & \text{on } \Gamma, \\
\quad p &= 0 & & \text{on } \Gamma_{in},
\end{aligned}
\]  

(1.1)

where $\Gamma_{in}$ is the inflow boundary, that is, for the unit outward normal vector $n$ on $\Gamma$:

\[\Gamma_{in} = \{(x, y) \in \Gamma | \beta \cdot n < 0\}.\]
The symbols $\Delta$, $\nabla$, and $\nabla \cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta \mathbf{u}$ is the vector of components $\Delta u_i$; $\mathbf{\beta} = (\beta_1, \beta_2)$, with $\beta_1 \geq C_0 > 0$ as a given $C^1$ function describing the given ambient flow. The number $\mu$ is a viscous constant; the number $\nu$ is a bulk viscous constant; $\mathbf{f} \in H^{-1}(\Omega) \times H^{-1}(\Omega)$ and $g \in L^2(\Omega)$ are vector and scalar functions, respectively. We assume that the Reynold number $\lambda := 1/\mu$ is in a compact interval $\Lambda = [\lambda_1, \lambda_2]$ of $\mathbb{R}^+$, that is, $0 < 1/\lambda_2 \leq \mu \leq 1/\lambda_1$ and we further assume that $\nu > 0$.

The finite element approach for the compressible Stokes equations have been recently reported in [2–5], for example, but not much for the compressible Navier-Stokes equations. In particular, in the course of solving nonlinear equations, one may employ Newton’s and Newton-like methods combined with other methods (see, for example, [6 – 8]). Such methods may be applied to the compressible Navier-Stokes equations (1.1). In this article we derive an equivalent iterative method ([9]) to Newton’s method following the techniques ([10]) in section II. Further, we discuss the finite element discretization error and regularity of the solution to the compressible Navier-Stokes equations (1.1) generated by Newton’s algorithm, which can be done by using the introduced equivalent iterative method. The equivalent iterative method can be described roughly as follows.

First, solve the momentum and continuity equations:

$$
\begin{align*}
-\mu \Delta \mathbf{u}^{(0)} - \nu \nabla \cdot \mathbf{u}^{(0)} + \nabla p^{(0)} &= \mathbf{f} \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u}^{(0)} + \mathbf{\beta} \cdot \nabla p^{(0)} &= g \quad \text{in } \Omega.
\end{align*}
$$

Second, solve the iterative scheme for the momentum and continuity equations of the form,

$$
\begin{align*}
-\mu \Delta \mathbf{u}^{(j)} - \nu \nabla \cdot \mathbf{u}^{(j)} + (\mathbf{z}^{(j-1)} \cdot \nabla)\mathbf{u}^{(j)} + (\mathbf{u}^{(j)} \cdot \nabla)\mathbf{z}^{(j-1)} + \nabla p &= -\mathbf{h}^{(j-1)} \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u}^{(j)} + \mathbf{\beta} \cdot \nabla p^{(j)} &= 0 \quad \text{in } \Omega,
\end{align*}
$$

where $\mathbf{z}^{(j-1)}$ and $\mathbf{h}^{(j-1)}$ are some known functions. Finally, the resulting approximation solution for (1.1) is the sum of solutions at each iteration step

$$
\mathbf{u}_n = \sum_{j=0}^{n} \mathbf{u}^{(j)} \quad \text{and} \quad p_n = \sum_{j=0}^{n} p^{(j)}.
$$

We will show that this is exactly the solution to Newton’s method in section II. We show that the Newton’s solution satisfies a $H^1$ regularity for a large $\mu$ by analyzing the regularity for the equivalent iterative method in section III. That is, we have the following: if $\|\mathbf{f}\|_{\nu-1}^2 + \|g\|^2 \leq 1$, then the solution $(\mathbf{u}_n, p_n)$ by Newton’s method (2.9) satisfies

$$
\|\mathbf{u}_n\|^2 + \|p_n\|^2 + \|\mathbf{\beta} \cdot \nabla p_n\|^2 \leq C_3 \quad \text{for all } n = 0, 1, \ldots .
$$

The finite element discretization error of the solution to (1.1) at each Newton’s iteration step is discussed in section III.

In this article, we use standard notations and definitions for the Sobolev spaces $H^k(\Omega)$, $L^2(\Omega)$, and $H^{1/2}(\Gamma)$, associated with proper inner products and norms. The subspace $H^0_0(\Omega) \subset H^1(\Omega)$ is used, which consists of the functions vanishing on the boundaries, and the dual space $H^{-1}(\Omega)$ is used with norm defined by
Define the product spaces $V := H_0^1(\Omega) \times H_0^1(\Omega)$ and the dual space $V^*$ of the space $V$ as $H^{-1}(\Omega) \times H^{-1}(\Omega)$ with the standard product norm. In the sequel, we use the boldface notation for the vector valued function or space and superscript * notation for the dual space. Let $\mathcal{M}$ be $L^2(\Omega)$ and define the space $\Theta \subset \mathcal{M}$ by

$$\Theta = \{ q \in \mathcal{M} | \beta \cdot \nabla q \in \mathcal{M}, q = 0 \text{ on } \Gamma_{in} \}.$$

II. NEWTON’S METHOD

The Newton’s iterative algorithm for the compressible Navier-Stokes equations (1.1) can be obtained by considering the following compressible Stokes equations:

$$\begin{cases}
-\Delta \mathbf{u} - \frac{\nu}{\mu} \nabla \cdot \mathbf{u} + \nabla p = \frac{1}{\mu} \mathbf{f} & \text{in } \Omega, \\
\nabla \cdot \mathbf{u} + \mu \beta \cdot \nabla p = g & \text{in } \Omega, \\
\mathbf{u} = 0 & \text{on } \Gamma, \\
p = 0 & \text{on } \Gamma_{in}. 
\end{cases} \quad (2.1)$$

The derivation of Newton’s algorithm will be done by introducing a linear solution operator $T$ for (2.1) and a $C^\infty$ operator $G$ for the compressible Stokes equations (1.1) (see [6]). For this purpose, we define the following function spaces

$$X := V \times \mathcal{M} \quad (2.2)$$

and

$$Y := (V^* \times L^2(\Omega)) \times H^{1/2}(\Gamma)^2. \quad (2.3)$$

Let $\mathcal{U} = (\mathbf{u}, p)$ and $\mathcal{V} = (v, q)$. Define the solution operator $T : Y \rightarrow X$ for (2.1) as $\mathcal{U} = T((\mathbf{f}, g), 0) \in X$ for all $((\mathbf{f}, g), 0) \in Y$ which means that $\mathcal{U} = (\mathbf{u}, p)$ is the solution of the compressible Stokes equations (2.1) for given functions $((\mathbf{f}, g), 0) \in Y$. In fact, this linear operator $T$ is well defined for a large $\mu$, since it can be shown that there exists a unique solution (see Proposition 3.3). Once $T$ is known, the operator $G$

$$G : \Lambda \times X \rightarrow Y, \quad (2.4)$$

is then given by

$$G(\lambda, \mathcal{U}) = ((\lambda(\mathbf{u} \cdot \nabla)\mathbf{u} - \lambda \mathbf{f}, -g), 0), \quad \forall \lambda \in \Lambda, \mathcal{U} \in X. \quad (2.5)$$

Hence, the nonlinear compressible Navier-Stokes equations (1.1) can be written as
where the vector form \( \mathcal{U} = (u, p) \) represents the solution pair for (1.1).

We assume that \( ((\lambda, \mathcal{U}(\lambda)); \lambda \in \Lambda) \) is a branch of nonsingular solutions of equation (2.6) (see p. 297 in [10]), so that the compressible Navier-Stokes Equations (1.1) has a unique solution \((u, p) \in X\) for given \((f, g) \in Y\).

**Lemma 2.1.** \((u, p) \in X\) is a solution of problem (1.1) if and only if \((u, \lambda p)\) is a solution of \(F(\lambda, \mathcal{U}) = 0\).

**Proof.** Observe that (1.1) may be equivalently rewritten as

\[
\begin{align*}
-\Delta u - \lambda \nabla \nabla \cdot u + \nabla (\lambda p) &= \lambda (f - (u \cdot \nabla)u) \quad \text{in } \Omega, \\
\nabla \cdot u + \mu \beta \cdot \nabla (\lambda p) &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma, \\
p &= 0 \quad \text{on } \Gamma_{in}.
\end{align*}
\]  

(2.7)

Hence, by definition of \(T\)

\[
(u, \lambda p) = T[((\lambda (f - (u \cdot \nabla)u), g), 0)]
\]

\[= -T \cdot G(\lambda, (u, \lambda p)).\]

This completes the proof. \(\blacksquare\)

Lemma 2.1 and (2.6) allow us to apply conventional abstract theory to obtain the Newton’s algorithm

\[
\mathcal{U}_{n+1} = \mathcal{U}_n - D_\mathcal{U}F(\lambda, \mathcal{U}_n)^{-1} \cdot F(\lambda, \mathcal{U}_n), \quad n = 0, 1, \ldots ,
\]

(2.8)

where \(D_\mathcal{U}F\) indicates the first Fréchet-derivative for \(F(\lambda, \mathcal{U})\). Then Newton’s algorithm (2.8) for (1.1) reads as follows:

Find \(\mathcal{U}_{n+1} = (u_{n+1}, p_{n+1}) \in X\) such that

\[
\begin{align*}
-\mu \Delta u_{n+1} - \nu \nabla \nabla \cdot u_{n+1} + (u_n \cdot \nabla)u_{n+1} + (u_{n+1} \cdot \nabla)u_n + \nabla p_{n+1} &= (u_n \cdot \nabla)u_n + f \quad \text{in } \Omega, \\
\nabla \cdot u_{n+1} + \mu \beta \cdot \nabla p_{n+1} &= g \quad \text{in } \Omega, \\
u u_{n+1} &= 0 \quad \text{on } \Gamma, \\
p &= 0 \quad \text{on } \Gamma_{in}.
\end{align*}
\]

(2.9)

The above derivation may imply that we can choose an initial guess \(\mathcal{U}^{(0)} = (u^{(0)}, p^{(0)}) \in X\) as the solution of (2.10). Hence we rewrite (2.9) as an iterative scheme (2.10)–(2.11) and show that (2.10)–(2.11) is in fact Newton’s algorithm.

First, solve

\[
\begin{align*}
-\mu \Delta u^{(0)} - \nu \nabla \nabla \cdot u^{(0)} + \nabla p^{(0)} &= f \quad \text{in } \Omega, \\
\nabla \cdot u^{(0)} + \mu \beta \cdot \nabla p^{(0)} &= g \quad \text{in } \Omega, \\
u u^{(0)} &= 0 \quad \text{on } \Gamma, \\
p^{(0)} &= 0 \quad \text{on } \Gamma_{in}.
\end{align*}
\]

(2.10)
With the solution $\mathcal{U}^{(0)}$ of the Stokes equation (2.10) as the initial guess, proceed the following iterative scheme:

Find $\mathcal{U}^{(j)} = (u^{(j)}, p^{(j)}) \in \mathbf{X}$ such that

\[
\begin{aligned}
-\mu \Delta u^{(j)} - \nu \nabla \cdot u^{(j)} + \left( \sum_{i=0}^{j-1} u^{(i)} \cdot \nabla \right) u^{(j)} + (u^{(j)} \cdot \nabla) \sum_{i=0}^{j-1} u^{(i)} + \nabla p^{(j)} &= - (u^{(j-1)} \cdot \nabla) u^{(j-1)} \quad \text{in } \Omega, \\
\nabla \cdot u^{(j)} + \beta \cdot \nabla p^{(j)} &= 0 \quad \text{in } \Omega, \\
p^{(j)} &= 0 \quad \text{on } \Gamma, \\
\nabla \cdot u^{(j)} + \beta \cdot \nabla p^{(j)} &= 0 \quad \text{on } \Gamma_{in}.
\end{aligned}
\]  
\hspace*{1cm} (2.11)

Now let

\[
\mathcal{U}_n = (u_n, p_n) := \mathcal{U}^{(0)} + \mathcal{U}^{(1)} + \cdots + \mathcal{U}^{(n)},
\]  
\hspace*{1cm} (2.12)

where

\[
u_n = \sum_{j=0}^{n} u^{(j)} \quad \text{and} \quad p_n = \sum_{j=0}^{n} p^{(j)}.\]  
\hspace*{1cm} (2.13)

This will be shown as the same solution to (2.9). The following lemma is immediate.

**Lemma 2.2.** We have

\[
\left( \sum_{i=0}^{n} a_i \right) \left( \sum_{i=0}^{n+1} b_i \right) = \sum_{i=1}^{n+1} \left( \sum_{j=0}^{i-1} a_j \right) b_i + \sum_{i=1}^{n} a_i \left( \sum_{j=0}^{i-1} b_j \right) + \sum_{i=1}^{n+1} a_{i-1} b_{i-1}.
\]  
\hspace*{1cm} \[\square\]

**Theorem 2.3.** The approximation solution $\mathcal{U}_n$ by the iterative scheme (2.11) is the solution generated by Newton’s algorithm (2.9).

**Proof.** First summing (2.11) from $j = 1$ to $n + 1$, and then adding (2.10), we have

\[
\begin{aligned}
-\mu \Delta \left[ \sum_{j=0}^{n+1} u^{(j)} \right] - \nu \nabla \cdot \left[ \sum_{j=0}^{n+1} u^{(j)} \right] + \sum_{j=1}^{n+1} \left( \sum_{i=0}^{j-1} u^{(i)} \cdot \nabla \right) u^{(j)} + (u^{(j)} \cdot \nabla) \sum_{i=0}^{j-1} u^{(i)} \\
+ \nabla \left[ \sum_{j=0}^{n+1} p^{(j)} \right] &= f - \sum_{j=1}^{n+1} \left[ (u^{(j-1)} \cdot \nabla) u^{(j-1)} \right], \\
\nabla \cdot \left[ \sum_{j=0}^{n+1} u^{(j)} \right] + \beta \cdot \nabla \left[ \sum_{j=0}^{n+1} p^{(j)} \right] &= g.
\end{aligned}
\]  
\hspace*{1cm} (2.14)
Let us rewrite (2.14) as the Newton’s iteration form

\[
\begin{cases}
- \mu \Delta u_{n+1} - \nu \nabla \cdot u_{n+1} + (u_n \cdot \nabla) u_{n+1} + (u_{n+1} \cdot \nabla) u_n \\
+ \nabla p_{n+1} = (u_n \cdot \nabla) u_n + f + \epsilon(n) & \text{in } \Omega, \\
\nabla \cdot u_{n+1} + \beta \cdot \nabla p_{n+1} = g & \text{in } \Omega,
\end{cases}
\]

(2.15)

where

\[
\epsilon(n) = (u_n \cdot \nabla) u_{n+1} + (u_{n+1} \cdot \nabla) u_n - \sum_{j=1}^{n+1} [(u^{(j)} \cdot \nabla) u_{j-1} + (u_{j-1} \cdot \nabla) u^{(j)}] \\
- \sum_{j=1}^{n+1} (u^{(j-1)} \cdot \nabla) u^{(j-1)} - (u_n \cdot \nabla) u_n.
\]

Using Lemma 2.2, one can verify easily that

\[\epsilon(n) = 0.\]

This completes the assertion of theorem.

**III. FINITE ELEMENT APPROXIMATION**

In this section, we discuss the regularity and the finite element approximation for the solution generated by Newton’s iteration (2.9). This can be done by considering the following linearized compressible Stokes equations defined on \( \Omega \subset \mathbb{R}^2 \) given by

\[
\begin{cases}
- \mu \nabla u - \nu \nabla \cdot u + (z \cdot \nabla) u + (u \cdot \nabla) z + \nabla p = h & \text{in } \Omega, \\
\nabla \cdot u + \beta \cdot \nabla p = g & \text{in } \Omega,
\end{cases}
\]

(3.1)

with

\[
\begin{cases}
u = 0 & \text{on } \Gamma, \\
p = 0 & \text{on } \Gamma_{in},
\end{cases}
\]

(3.2)

where \( z \in V \) is a known bounded function and \( h \in V^* \) and \( g \in M \). We derive finite element discretization error for (2.9) and estimate the \( H^1 \) regularity of (3.1) following similar arguments in [2]. Then we apply such a regularity result to (2.10) and (2.11), and finally we get the regularity result for (2.9). Define the bilinear form \( a(\cdot, \cdot ; z) \) as, for all \( u, v \in V \),

\[
a(u, v; z) = \int_{\Omega} \mu \nabla u \cdot \nabla v + \nu \nabla \cdot (u \nabla v) + ((z \cdot \nabla) u + (u \cdot \nabla) z) v d\Omega,
\]

and the bilinear forms \( b(\cdot, \cdot) \), \( c(\cdot, \cdot) \) and \( d(\cdot, \cdot) \) as
\[
\begin{align*}
\mathbf{b}(\mathbf{v}, p) &= - \int_\Omega p(\nabla \cdot \mathbf{v}) d\Omega, \quad \mathbf{v} \in \mathbf{V}, \quad p \in \mathcal{M}, \\
\mathbf{c}(p, q) &= \int_\Omega (\mathbf{\beta} \cdot \nabla p)(\mathbf{\beta} \cdot \nabla q) d\Omega, \quad p, q \in \mathcal{Q}, \\
\mathbf{d}(\mathbf{u}, q) &= \int_\Omega (\nabla \cdot \mathbf{u})(\mathbf{\beta} \cdot \nabla q) d\Omega, \quad \mathbf{u} \in \mathbf{V}, \quad q \in \mathcal{Q}.
\end{align*}
\]

Define the linear functionals \( \langle \mathbf{h}, \cdot \rangle \) and \( \langle g, \cdot \rangle \) as
\[
\langle \mathbf{h}, \mathbf{v} \rangle = \int_\Omega \mathbf{h} \cdot \mathbf{v} d\Omega, \quad \langle g, q \rangle = \int_\Omega g(\mathbf{\beta} \cdot \nabla q) d\Omega, \quad \mathbf{v} \in \mathbf{V}, q \in \mathcal{Q}.
\]

Then the weak formulation corresponding to (3.1) and (3.2) can be given by the following: Find \((\mathbf{u}, p) \in \mathbf{V} \times \mathcal{Q}\) such that
\[
\begin{cases}
\mathbf{a}(\mathbf{u}, \mathbf{v}; \mathbf{z}) + \mathbf{b}(\mathbf{v}, p) = \langle \mathbf{h}, \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{V}, \\
\mathbf{c}(p, q) + \mathbf{d}(\mathbf{u}, q) = \langle g, q \rangle & \forall q \in \mathcal{Q}.
\end{cases}
\] (3.3)

We formulate the finite element approximation for (3.3) corresponding to (2.9) and establish an error analysis for this approximation. For this purpose, the finite dimensional subspaces \( \mathbf{V}_h \subseteq \mathbf{V} \) and \( \mathcal{Q}_h \subseteq \mathcal{Q} \) will be used. Let \( \mathcal{T}_h \) be a family of triangulations of \( \Omega \) by standard finite element subdivisions of \( \Omega \) into quasi-uniform triangles with \( h = \max\{diam(K) : K \in \mathcal{T}_h\} \). Near a curved portion of the boundary \( \Gamma \), we use isoparametric triangles with one curved side. It is assumed that the curved elements in the triangulations of the domain of \( \Omega \) not to be "too curved," so that the standard interpolation error on a straight triangle will hold for a curved triangle (see [11–13]).

Assume that the finite element spaces \( \mathbf{V}_h \) and \( \mathcal{Q}_h \) satisfy the following approximation properties:

\[
\inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 \leq C h^k \|\mathbf{u}\|_{k + 1}, \quad \forall \mathbf{v} \in \mathbf{V} \cap \mathcal{H}^{k+1/2}(\Omega), \tag{3.4}
\]

\[
\inf_{\mathbf{q} \in \mathcal{Q}_h} \|p - \mathbf{q}\|_1 \leq C h^{m+1} \|p\|_{m+1}, \quad \forall p \in \mathcal{Q} \cap \mathcal{H}^{m+1}(\Omega), \tag{3.5}
\]

where \( C \) is a constant independent of \( h \).

Then the finite element method corresponding to (3.3) is as follows:

Find \((\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathcal{Q}_h\) such that
\[
\begin{cases}
\mathbf{a}(\mathbf{u}_h, \mathbf{v}_h; \mathbf{z}) + \mathbf{b}(\mathbf{v}_h, p_h) = \langle \mathbf{h}, \mathbf{v}_h \rangle & \forall \mathbf{v}_h \in \mathbf{V}_h, \\
\mathbf{c}(p_h, q_h) + \mathbf{d}(\mathbf{u}_h, q_h) = \langle g, q_h \rangle & \forall q_h \in \mathcal{Q}_h.
\end{cases}
\] (3.6)
We assume that \( b(\cdot, \cdot) \) satisfies the inf-sup condition on \( V_h \times \mathcal{Q}_h \); there is a constant \( \alpha > 0 \) such that

\[
\inf_{q_h \in \mathcal{Q}_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_1 \|q_h\|} \geq \alpha. \tag{3.7}
\]

Recall the Poincaré constant such that

\[
C_p \|u\|_1^2 \leq \|\nabla u\|^2, \quad \forall u \in V, \tag{3.8}
\]

where \( C_p \) is a positive constant. Let, for some positive constant \( B \),

\[
V_1 := \{ z \in V \|z\|_1 \leq B \}. \tag{3.9}
\]

**Lemma 3.1.** For any \( u, v \in H^1_0(\Omega)^2 \) and \( z \in V_1 \), the following holds:

\[
\|((z \cdot \nabla)u, v) + ((u \cdot \nabla)z, v)\| \leq C_T \|u\|_1 \|v\|_1, \tag{3.10}
\]

where \( C_T \) is a constant dependent on \( B \).

**Proof.** First recall the following result such that

\[
\left| \int_\Omega D_i uvwd\Omega \right| \leq \tilde{C} \|u\|_1 \|v\|_1 \|w\|_1, \quad i = 1, 2
\]

for all \( u, v, \) and \( w \in H^1(\Omega) \), where \( D_i \) denotes the derivative with respect to the \( i \)th coordinate variable in \( \mathbb{R}^2 \) and \( \tilde{C} \) is a positive constant. This result can be found in, for example, Lemma 7 in [14]. Hence one may have the conclusion using this result.

We recall from [5] that for the domain \( \Omega \) with the \( C^2 \) boundary, there is some constant \( C_\beta > 0 \) such that

\[
\|p\| \leq C_\beta \|\nabla p\|, \quad \forall p \in \mathcal{Q}. \tag{3.11}
\]

Now, we have the continuity and coercivity for the bilinear form \( a(\cdot, \cdot; z) \) as the following.

**Lemma 3.2.** Let \( z \in V_1 \). Then for a sufficiently large \( \mu \) we have

\[
\mu_1 \|u\|_1^2 \leq a(u, u; z), \quad \forall u \in V, \tag{3.12}
\]

where \( \mu_1 := \mu C_p - C_T > 0 \) and

\[
a(u, v; z) \leq \tilde{C}_1 \|u\|_1 \|v\|_1, \quad \forall u, v \in V, \tag{3.13}
\]

where \( \tilde{C}_1 \) is a positive constant depending on \( \mu, v, \) and \( C_T \).
Note that we can take \( \mu_1 := \mu C_p - C_T \) as a positive constant for a sufficiently large \( \mu \). Hence we have the first inequality. The second inequality is easily verified by using Schwarz inequality and (3.10).

We first discuss the \( H^1 \) regularity of the solution to the compressible Navier-Stokes equations generated by Newton’s method. This can be done by studying regularity of (3.1).

**Proposition 3.3.** Let \( h \in V^* \) and \( g \in M \). Assume that \( \mu \) is large enough and \( z \in V_1 \). Then the solution \( (u, p) \in V \times \Omega \) of the weak form (3.3) satisfies a priori estimate

\[
\|u\|^2 + \|p\|^2 + \|\beta \cdot \nabla p\|^2 \leq C_2(\|h\|^2 + \|g\|^2),
\]

(3.14)

where \( C_2 \) is a positive constant depending on \( C_1 \) and \( C_\beta \). In particular,

\[
\|u\|^2 \leq \frac{K_1}{\mu_1^2}(\|h\|^2 + \|g\|^2),
\]

(3.15)

where \( K_1 \) is a positive constant independent of \( \mu_1 \), which can be chosen sufficiently large.

**Proof.** Since the weak solution \( (u, p) \in V \times \Omega \) satisfies

\[
\begin{align*}
\{ a(u, u; z) + b(u, p) &= (h, u) \\
c(p, p) + d(u, p) &= (g, p),
\end{align*}
\]

(3.16)

we have, from the second equation of (3.16),

\[
\|\beta \cdot \nabla p\|^2 = c(p, p) \leq |(g, p)| + |d(u, p)| \leq \|g\| \|\beta \cdot \nabla p\| + \|\nabla \cdot u\| \|\beta \cdot \nabla p\|
\]

\[
\leq (\|g\| + \|u\|) \|\beta \cdot \nabla p\|, \quad \text{for all } p \in \Omega
\]

and

\[
\|\beta \cdot \nabla p\|^2 \leq 2(\|g\|^2 + \|u\|^2), \quad \forall p \in \Omega.
\]

(3.17)

Moreover, from (3.11) and (3.17), we have

\[
\|p\|^2 \leq C_1^2 \|\beta \cdot \nabla p\|^2 \leq 2C_1^2(\|g\|^2 + \|u\|^2), \quad \forall p \in \Omega.
\]

(3.18)

It follows, from the first equation of (3.16), Lemma 3.2 and the \( \varepsilon \)-inequality, that
\[ \mu_1 \| \mathbf{u} \|^2 \leq a(\mathbf{u}, \mathbf{u}; \mathbf{z}) - b(\mathbf{u}, p) \leq \| \mathbf{h} \|_1 \| \mathbf{u} \|, \]
\[ + \| \mathbf{u} \|_1 \| p \| \leq \frac{1}{2 \varepsilon_1} \| \mathbf{h} \|_1^2 + \frac{1}{2 \varepsilon_2} \| p \|^2 + \left( \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2} \right) \| \mathbf{u} \|^2. \]

By taking \( \varepsilon_1 = \varepsilon_2 = \mu_1 / 2 \), we have
\[ \| \mathbf{u} \|^2 \leq \frac{2}{\mu_1} \| \mathbf{h} \|_1^2 + \frac{2}{\mu_1} \| p \|^2. \] (3.19)

Because we can choose \( \mu_1 = \mu C_p - C_T > 0 \) for large \( \mu \), we may have, by combining (3.18) and (3.19),
\[ \| \mathbf{u} \|^2 \leq K_1 \frac{1}{\mu_1} (\| \mathbf{h} \|_1^2 + \| g \|^2), \] (3.20)

where \( K_1 \) is a constant depending only on \( C_\beta \). Now, using (3.17), (3.18), and (3.20), one may find a positive constant \( C_2 \) depending on \( C_\beta \) and \( C_1 \) such that
\[ \| \mathbf{u} \|^2 + \| p \|^2 + \| \mathbf{\beta} \cdot \nabla p \|^2 \leq C_2 (\| \mathbf{h} \|_1^2 + \| g \|^2). \] (3.21)

This yields the conclusion.

Let us consider the regularity of the solution to (2.9) at each iteration step. In the rest of this section, without loss of generality we may assume that
\[ \| f \|_1^2 + \| g \|^2 \leq 1. \] (3.22)

**Theorem 3.4.** Assume that (3.22) holds. Under the assumptions in Proposition 3.3, the solution \( (\mathbf{u}_n, p_n) \) of (2.9) satisfies
\[ \| \mathbf{u}_n \|^2 + \| p_n \|^2 + \| \mathbf{\beta} \cdot \nabla p_n \|^2 \leq C_3 \quad \forall n = 0, 1, \ldots, \] (3.23)

where \( C_3 \) is a positive constant independent of \( n \).

**Proof.** At first, let us consider the simplified compressible Stokes equations (2.10), that is, \( \mathbf{u} = \mathbf{u}^{(0)}, p = p^{(0)}, \mathbf{z} = \mathbf{0} \) with \( \mathbf{h} = \mathbf{f} \) in (3.1). Then, from Proposition 3.3 with (3.22),
\[ \| \mathbf{u}^{(0)} \|^2 \leq K_1 \frac{1}{\mu_1} (\| \mathbf{f} \|_1^2 + \| g \|^2) \leq \frac{K_1}{\mu_1} =: \gamma \] (3.24)

and
\[ \| \mathbf{u}^{(0)} \|^2 + \| p^{(0)} \|^2 + \| \mathbf{\beta} \cdot \nabla p^{(0)} \|^2 \leq C_2 (\| \mathbf{f} \|_1^2 + \| g \|^2) \leq C_2, \] (3.25)

where \( \gamma \) can be made less than 1, which is possible by taking a large \( \mu \) so that \( \mu_1 \) is large.

Now, we will show that the solution \( \mathbf{u}^{(j+1)} \) of (2.11) satisfy
by mathematical induction. At the 1st iteration process in (2.11), we can take
\[ z = u^{(0)} \in V_1 \]
and
\[ u = u^{(1)}, \quad p = p^{(1)}, \quad h = -(u^{(0)} \cdot \nabla)u^{(0)}, \quad \text{and} \quad g = 0 \]
in (3.1). Then, one may have from (3.15) and (3.24)
\[ \|u^{(1)}\|_1^2 \leq \gamma\|(u^{(0)} \cdot \nabla)u^{(0)}\|_{-1}^2 \leq \gamma\|u^{(0)}\|_1^4 \leq \gamma^3. \]  
(3.26)

At the \( j \)th iteration process, by taking
\[ u = u^{(j)}, \quad p = p^{(j)}, \quad z = \sum_{i=0}^{j-1} u^{(i)}, \quad f = -(u^{(j-1)} \cdot \nabla)u^{(j-1)}, \quad \text{and} \quad g = 0 \]
in (3.1), assume that we have
\[ \|u^{(j)}\|_1^2 \leq \gamma^{2^{j+1}-1}. \]  
(3.27)

Then, for the \((j + 1)\)th iteration process, consider (3.1) by taking
\[ u = u^{(j+1)}, \quad p = p^{(j+1)}, \quad z = \sum_{i=0}^{j} u^{(i)}, \quad f = -(u^{(j)} \cdot \nabla)u^{(j)}, \quad \text{and} \quad g = 0. \]

Therefore, using (3.27), it follows that
\[ \|z\|_1 \leq \sum_{i=0}^{j} \|u^{(i)}\|_1 \leq \sum_{i=0}^{j} [\gamma^{2^{i+1}-1}]^{1/2} \leq \frac{\gamma^{1/2}}{1 - \gamma^{1/2}}. \]  
(3.28)

Hence
\[ z = \sum_{i=0}^{j} u^{(i)} \in V_1 \]
satisfies Lemma 3.2, so that (3.15) in Proposition 3.3 yields
\[ \|u^{(j+1)}\|_1^2 \leq \gamma\|(u^{(j)} \cdot \nabla)u^{(j)}\|_{-1}^2 \leq \gamma\|u^{(j)}\|_1^4 \leq \gamma(\gamma^{2^{j+1}-1})^2 = \gamma^{2^{j+2}-1}, \]  
(3.29)
which completes the induction arguments.

In particular, again Proposition 3.3 applied to (2.11) yields

\[
\|u^{j+1}\|^2 + \|p^{j+1}\|^2 + \|\beta \cdot \nabla p^{j+1}\|^2 \leq C_2(\|u^j\|^2 + \|\beta \cdot \nabla u^j\|^2)^2 \leq C_3 \gamma^{2j+2},
\]

where \(j = 0, 1, \ldots, n - 1\). Thus, we conclude from (3.25) and (3.30) that

\[
\|u_n\|^2 + \|p_n\|^2 + \|\beta \cdot \nabla r_n\|^2 \leq \sum_{j=0}^{n} (\|u^j\|^2 + \|p^j\|^2 + \|\beta \cdot \nabla r^j\|^2) \leq C_2 \gamma^{2n+2} < C_2 \frac{\gamma}{1 - \gamma} =: C_3.
\]

\[
(3.30)
\]

We can now show the finite element discretization error for the solution generated by Newton’s method. We note that if (3.7) and (3.22) is satisfied, then the finite element approximation problem (3.6) has at least one solution. By taking

\[
u = u_{n+1}, \quad p = p_{n+1}, \quad z = u_n, \quad \text{and} \quad h = (u_n \cdot \nabla)u_n + f
\]

in (3.1), we will show the following error estimate for the finite element approximate solution generated by Newton’s method (2.9). According to Theorem 3.4, note that \(u_n \in V_h\).

\textbf{Theorem 3.5.} Under the assumption of Theorem 3.4, the weak solution \((u_{n+1}, p_{n+1}) \in V \times \Omega\) by Newton’s method of (3.3) and its approximate solution \((u_{n+1,h}, p_{n+1,h}) \in V_h \times \Omega_h\) of (3.6) by finite element methods satisfy the following error estimate: there is a constant \(C = C(\mu, \beta, \Omega)\) such that

\[
\|u_{n+1} - u_{n+1,h}\| + \|p_{n+1} - p_{n+1,h}\| + \|\beta \cdot \nabla (p_{n+1} - p_{n+1,h})\| \\
\leq C \inf \{\|u_{n+1} - v_h\| + \|p_{n+1} - q_h\| + \|\beta \cdot \nabla (p_{n+1} - q_h)\|\},
\]

where the infimum is taken over all \(v_h \in V_h\) and \(q_h \in \Omega_h\).

\textbf{Proof.} For any \(v_h, \tilde{v}_h \in V_h\) and \(\tilde{q}_h \in \Omega_h\), we have

\[
\left\{
\begin{align*}
& a(u_{n+1} - u_{n+1,h}, \tilde{v}_h; z) + b(\tilde{v}_h, p_{n+1} - p_{n+1,h}) = 0, \\
& c(p_{n+1} - p_{n+1,h}, \tilde{q}_h) + d(u_{n+1} - u_{n+1,h}, \tilde{q}_h) = 0.
\end{align*}
\right.
\]

(3.32)

Hence, for any \(v_h, \tilde{v}_h \in V_h\), and \(q_h, \tilde{q}_h \in \Omega_h\), we have

\[
\left\{
\begin{align*}
& a(u_{n+1} - v_h, \tilde{v}_h; z) + b(\tilde{v}_h, p_{n+1} - p_{n+1,h}) = a(u_{n+1,h} - v_h, \tilde{v}_h; z), \\
& c(p_{n+1} - q_h, \tilde{q}_h) + d(u_{n+1} - u_{n+1,h}, \tilde{q}_h) = c(p_{n+1,h} - q_h, \tilde{q}_h).
\end{align*}
\right.
\]

(3.33)

Then by taking \(\tilde{q}_h = p_{n+1,h} - q_h\) into the second equation of (3.33), we have

\[
\|\beta \cdot \nabla (p_{n+1} - q_h)\| \leq \|\beta \cdot \nabla (p_{n+1} - p_{n+1,h})\| + \|\beta \cdot \nabla (p_{n+1,h} - q_h)\| \\
\leq \|\beta \cdot \nabla (p_{n+1} - q_h)\| + \|\beta \cdot \nabla (p_{n+1,h} - q_h)\|.
\]
Hence we have
\[\| \beta \cdot \nabla (p_{n+1} - q_h) \| \leq \| u_{n+1} - u_{n+1,h} \|_1 + \| \beta \cdot \nabla (p_{n+1} - q_h) \|,\]
and by triangle inequality,
\begin{align*}
\| \beta \cdot \nabla (p_{n+1} - p_{n+1,h}) \| &\leq \| \beta \cdot \nabla (p_{n+1} - q_h) \| + \| \beta \cdot \nabla (p_{n+1,h} - q_h) \| \\
&\leq \| u_{n+1} - u_{n+1,h} \|_1 + 2\| \beta \cdot \nabla (p_{n+1} - q_h) \|.
\end{align*}
(3.34)

Since \( p_{n+1} \in \mathcal{Q} \) by (3.23) and \( p_{n+1,h} \in \mathcal{Q}_h \subset \mathcal{Q} \), we have
\begin{align*}
\| p_{n+1} - p_{n+1,h} \| &\leq C_\beta \| \beta \cdot \nabla (p_{n+1} - p_{n+1,h}) \| \\
&\leq C_\beta \| u_{n+1} - u_{n+1,h} \|_1 + 2C_\beta \| \beta \cdot \nabla (p_{n+1} - q_h) \|
\end{align*}
(3.35)

Next by taking \( \tilde{v}_h = u_{n+1,h} - v_h \) into the first equation of (3.33) and Lemma 3.2, we have
\[C_1\| u_{n+1,h} - v_h \|_1 \leq a(u_{n+1,h} - v_h, u_{n+1,h} - v_h; z)
\]
\[= a(u_{n+1} - v_h, u_{n+1,h} - v_h; z) + b(u_{n+1,h} - v_h, p_{n+1} - p_{n+1,h})
\]
\[\leq C_4(\| u_{n+1} - v_h \|_1 \| u_{n+1,h} - v_h \|_1 + \| u_{n+1,h} - v_h \|_1 \| p_{n+1} - p_{n+1,h} \|),
\]
where \( C_4 \) is a positive constant depending on \( \tilde{C}_1 \). Hence we get
\[C_1\| u_{n+1,h} - v_h \|_1 \leq C_4(\| u_{n+1} - v_h \|_1 + \| p_{n+1} - p_{n+1,h} \|),
\]
and by triangle inequality,
\[C_1\| u_{n+1} - u_{n+1,h} \|_1 \leq C_1\| u_{n+1} - v_h \|_1 + C_1\| u_{n+1,h} - v_h \|_1 \leq C_5(\| u_{n+1} - v_h \|_1 + \| p_{n+1} - p_{n+1,h} \|),
\]
(3.36)

where \( C_5 \) is a positive constant depending on \( C_1 \) and \( C_4 \). Combining (3.34), (3.35) and (3.36), we have the conclusion. \( \blacksquare \)

Now, we give an error estimate for the finite element approximated solution of (3.6).

**Corollary.** Assume that the hypotheses of Theorem 3.4 hold and suppose that the solution of the weak form (3.3) by Newton’s method is in \([u_{n+1}, P_{n+1}] \in (V \times \mathbb{Q}) \cap (H^{k+1}(\Omega) \times H^m(\Omega))\). Then the finite element approximate solution \([u_{n+1,h}, P_{n+1,h}] \in (V_h \times \mathcal{Q}_h) \cap (H^{k+1}(\Omega) \times H^m(\Omega))\) of (3.6) satisfies the following:
\[\| u_{n+1} - u_{n+1,h} \|_1 + \| P_{n+1} - P_{n+1,h} \| + \| \beta \cdot \nabla (P_{n+1} - P_{n+1,h}) \| \leq C(h^{\alpha} \| u_{n+1} \|_{k+1} + h^{m-1} \| P_{n+1} \|_m),\]
where \( C \) is a constant.

**Proof.** It comes from (3.4) and (3.5) with Theorem 3.5. \( \blacksquare \)

The authors thank anonymous referees for many valuable corrections and comments.
References


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